Tutorial 12: Selected problems of Assignment 12

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Goal: Study the (absolute) convergence of $\sum_{n=1}^{\infty} x_n$ (assume $x_n \neq 0$)

The following suggests an order of applying different tests:

Step 0: n^{th} term test: $\lim_n x_n \neq 0 \Rightarrow$ diverges

Q1c) $x_n = 2^{-\frac{1}{n}}$: $\lim_n x_n = 1 \neq 0 \Rightarrow$ diverges

Q2d) $x_n = (-1)^n \frac{n}{n+1}$: $\lim_n x_{2n} = 1 \neq 0 \Rightarrow$ diverges

Step 1: Non-comparison tests:

(i) Root test: $\lim_n |x_n|^{\frac{1}{n}} < 1 \Rightarrow$ absolutely converges

$\quad$ — " — $> 1 \Rightarrow$ diverges

Q3b) $x_n = (\log n)^{-n}$; $|x_n|^{\frac{1}{n}} = \frac{1}{\log n} \to 0 \Rightarrow$ abs. conv.

Q4b) $x_n = n^n e^{-n}$; $|x_n|^{\frac{1}{n}} = n e^{-1} \to \infty \Rightarrow$ diverges

(ii) Ratio test: $\lim_n \left|\frac{x_{n+1}}{x_n}\right| < 1 \Rightarrow$ abs. conv.

$\quad$ (— " —) $> 1 \Rightarrow$ diverges

Q2c) $x_n = \frac{n!}{n^n}$; $\left|\frac{x_{n+1}}{x_n}\right| = \frac{1}{(1+\frac{1}{n})^{n+1}} \to \frac{1}{e} < 1 \Rightarrow$ abs. conv.

Q4e) $x_n = n! e^{-n}$; $\left|\frac{x_{n+1}}{x_n}\right| = \frac{n+1}{e} \to \infty \Rightarrow$ diverges.

(iii) Raabe's test: $\lim_n n\left(1-\left|\frac{x_{n+1}}{x_n}\right|\right) > 1 \Rightarrow$ abs. conv

$\left(\text{———} \;''\; \text{———}\right) < 1 \Rightarrow$ not abs. conv.

Q1a) $x_n = \frac{1}{(n+1)(n+2)}$; $\left|\frac{x_{n+1}}{x_n}\right| = \frac{n+1}{n+3} = 1 - \frac{2}{n+3}$;

$\lim_n n\cdot\left(1-\left|\frac{x_{n+1}}{x_n}\right|\right) = \lim_n \frac{2n}{n+3} = 2 > 1 \Rightarrow$ abs. conv.

(iv) Integral test: $\exists\, f:[K,+\infty)\to\mathbb{R}$ positive decreasing continuous

$\exists\, K\in\mathbb{N}$ such that $x_n = f(n),\ \forall n\geq K$

then $\sum_{n=1}^{\infty} x_n$ converges $\Leftrightarrow \int_K^{\infty} f$ exists

Q3e) $x_n = \frac{1}{n\log n}$; let $f:[2,+\infty)\to\mathbb{R}$ be $f(x) = \frac{1}{x\log x}$

$\therefore \int_2^{+\infty} f = \int_2^{+\infty} \frac{1}{x\log x}\,dx = \left[\log(\log x)\right]_2^{+\infty}$ does not exists

$\therefore \sum_{n=1}^{\infty} x_n$ diverges

Step 2) Comparison tests: assume $x_n \geq 0$.

- If $\exists\,(y_n)$ such that $0\leq x_n\leq y_n$ and $\sum_{n=1}^{\infty} y_n$ converges, then $\sum_{n=1}^{\infty} x_n$ converges
- If $\exists\,(z_n)$ such that $0\leq z_n\leq x_n$ and $\sum_{n=1}^{\infty} z_n$ diverges, then $\sum_{n=1}^{\infty} x_n$ diverges.

See solutions to (Q3c), (Q3d) as examples.